Appendix E
Speaker Coupling

A vibrating speaker cone in an "infinite baffle" performs roughly the same as a vibrating disk in an infinite half space. In fact, circular ports generally perform as if they were covered by a vibrating diaphragm (ignoring the higher order solutions).

Assume a disk diaphragm of radius a which vibrates in the z direction with velocity function $v(t)$. Define the mask function $u(r)$ such that

$$u(r) = \begin{cases} 1, & r < a \\ 0, & \text{otherwise} \end{cases} \tag{E1}$$

which has the transform

$$U(k_r) = \int_0^a r J_0(k_r r)\, dr = \frac{a}{k_r} J_1(a k_r) \tag{E2}$$

since

$$\int x J_0(x)\, dx = x J_1(x) + C \quad .$$

We need to find a wave equation solution which satisfies the boundary condition

$$\left.\frac{\partial p}{\partial z}\right|_{z=0} = -\rho \frac{\partial v}{\partial t} u(r) \tag{E3}$$

which comes from applying $(C2)$ to the surface containing the diaphragm. Using the wave equation solution from $(D6)$, we see that $(E3)$ is satisfied when

$$j k_z B_\omega J_0(k_r r) = \rho(j\omega v_\omega) u(r)$$

($j = \sqrt{-1}$) and, taking the transform of both sides w.r.t. r, we get

$$j k_z B_\omega = j\omega \rho v_\omega U(k_r) \quad .$$

Finally, substituting for $U(k_r)$ using $(E2)$, we see that

$$B_\omega = \frac{\rho\omega}{k_z} U(k_r) v_\omega = \frac{\rho\omega a}{k_r k_z} J_1(a k_r) v_\omega \tag{E4}$$

where $k_z^2 + k_r^2 = (\omega/c)^2$.

One can compute the pressure at any r and z by evaluating the inverse transform

$$p_\omega(r, z) = \int_0^\infty B_\omega J_0(k_r r) e^{jzk_z} k_r\, dk_r \tag{E5}$$

noting that k_z becomes imaginary when $k_r > \omega/c$.

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The ratio of the surface area of a disk of radius a to the surface area of a hemisphere of radius a is given by

$$\frac{\pi a^2}{2\pi a^2} = \frac{1}{2}$$

and the corresponding spherical wave as $a \to 0$, $(D9)$, must satisfy the boundary condition

$$\left.\frac{\partial}{\partial r} P\right|_{r=a} = -\rho \frac{\partial v}{\partial t}$$

and therefore

$$C_\omega \frac{j\omega a/c - 1}{a^2} e^{j\omega a/c} = j\omega\rho\frac{v_\omega}{2}$$

where v_ω is the velocity of the disk.

From this we see that

$$C_\omega \to \frac{-j\omega\rho}{2} v_\omega a^2 \quad \text{as} \quad a \to 0 \tag{E6}$$

and from (E4) we see that

$$B_\omega \to \frac{\rho\omega}{2k_z} v_\omega a^2 = \frac{\rho c}{2\cos\theta} v_\omega a^2 \quad \text{as} \quad a \to 0 \tag{E7}$$

since

$$J_1(ak_r) \to \frac{ak_r}{2} \quad \text{as} \quad a \to 0$$

and using $k_z = \frac{\omega}{c}\cos\theta$.

Comparing (E6) to (E7)we see that

$$C_\omega = \frac{-j\omega}{c}\cos\theta B_\omega \tag{E8}$$

which gives us the scale factor α in (D19) and (D20).

Far Field

Substituting for B_ω in (E8) using (E4) we have

$$C_\omega(\theta) = \frac{-j\omega^2\rho a\cos\theta}{ck_r k_z} J_1(ak_r) v_\omega$$

where

$$k_r = \frac{\omega}{c}\sin\theta \quad \text{and} \quad k_z = \frac{\omega}{c}\cos\theta$$

and therefore

$$C_\omega(\theta) = \frac{-j\rho ac}{\sin\theta} J_1(\omega a\sin\theta/c) v_\omega \quad . \tag{E9}$$

At $\theta = 0$ (directly in front of the speaker) this simplifies to

$$C_\omega(0) = \frac{-j\omega\rho a^2}{2} v_\omega \tag{E10}$$

since

$$\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2} \quad .$$

Substituting for $C\omega$ in (D18) using (E10) we see that the sound pressure at some distance d directly in front of the speaker is given by

$$p_\omega \to \left(\frac{-j\omega\rho a^2}{2}\right) \frac{e^{j\omega d/c}}{d} v_\omega \quad \text{as} \quad d \to \infty$$

and therefore the coupling factor $T(S)$ used in Appendix B is given by

$$T(S) = S\left(\frac{\rho a^2}{2}\right)\frac{e^{-Sd/c}}{d} \tag{E11}$$

where $S = -j\omega$.

Near Field

The last quantity of interest is the force experienced by the diaphragm, which can be expressed as

$$F_\omega = \int_0^a p_\omega(r,0) r\, dr \quad . \tag{E12}$$

Substituting for $p_\omega(r,0)$ using (E5) we get

$$\begin{aligned} F_\omega &= \int_0^a \left(\int_0^\infty B_\omega J_0(rk_r) k_r\, dk_r \right) r\, dr \\ &= \int_0^\infty B_\omega \int_0^a J_0(rk_r) r k_r\, dr\, dk_r \\ &= \int_0^\infty B_\omega a J_1(ak_r)\, dk_r \quad . \end{aligned}$$

Substituting for B_ω using (E4) gives us

$$F_\omega = \rho\omega a^2 v_\omega \int_0^\infty \frac{J_1^2(ak_r)}{k_r k_z}\, dk_r \tag{E13}$$

where

$$k_z = \sqrt{(\omega/c)^2 - k_r^2} \quad .$$

Note that the integral should be broken up as

$$\int_0^\infty \frac{J_1^2(ax)}{x\sqrt{k^2 - x^2}}\, dx = \int_0^k \frac{J_1^2(ax)}{x\sqrt{k^2 - x^2}}\, dx - j \int_k^\infty \frac{J_1^2(ax)}{x\sqrt{x^2 - k^2}}\, dx$$

where $k = \omega/c$.

The results of this integration can be expressed as [1, p. 383]

$$F_\omega = \pi a^2 \rho c \big(\theta_0(\gamma) - j\chi_0(\gamma)\big) v_\omega \tag{E14}$$

where $\gamma = 2ak = 2\omega a/c = 4\pi a/\lambda$, $\theta_0(\gamma)$ can be expressed as

$$\theta_0(\gamma) = 1 - 2\gamma J_1(\gamma) \tag{E15}$$

and one can obtain a power series expression for $\chi_0(\gamma)$. Alternatively, one can solve the differential equations

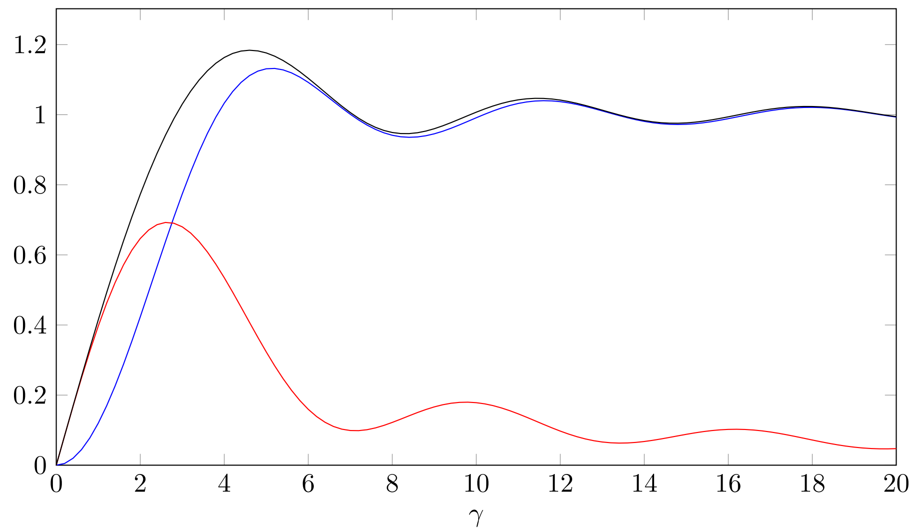
$$x\theta_0''(x) + 3\theta_0'(x) + x\theta_0(x) = x \tag{E16}$$

starting from $\theta_0(0) = 0$ and $\theta_0'(0) = 0$ and

$$x\chi_0''(x) + 3\chi_0'(x) + x\chi_0(x) = \frac{4}{\pi} \tag{E17}$$

starting from $\chi_0(0) = 0$ and $\chi_0'(0) = 4/(3\pi)$. This is more useful computationally since the power series solution doesn't converge well for large γ (e.g, $\gamma > 10$ or so).

$F(\gamma) = \theta_0(\gamma) - j\chi_0(\gamma)$



Finally, the load $Z(S)$ used in Appendix B can be expressed as

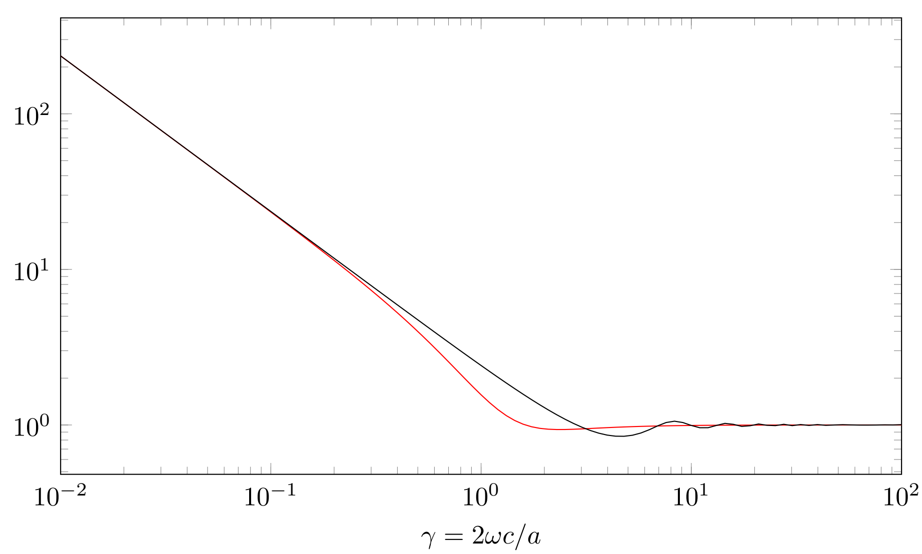
$$Z(S) = \frac{v_\omega}{F_\omega} = \left(\frac{1}{\pi a^2 \rho c}\right) \frac{\theta_0(\gamma) + j\chi_0(\gamma)}{\theta_0^2(\gamma) + \chi_0^2(\gamma)}$$

where $S = -j\omega = -j\gamma c/2a$. Based on asymptotic behavior[1, p. 384], one can approximate $Z(S)$ with the second order function

$$Z(S) \approx \left(\frac{1}{\pi a^2 \rho c}\right) \frac{S^2 + 1.79358(c/2a)S + 2.0(c/2a)^2}{S(S + 0.84883c/2a)} \quad . \qquad (E18)$$

The amplitude of $Z(\gamma)$ (when $\pi a^2 \rho c = 1$) is shown in the following graph by the black line and the approximation by the red line.

Air Loading $Z(\gamma)$



References

[1] Phillip M. Morse and K. Uno Ingard, **Theoretical Acoustics**, ISBN 0-691-02401-4