Appendix A
Lumped Parameter Systems

The tools developed for analysis of electrical circuits can also be applied to mechanical and acoustic systems.

$\mathcal{V}$ is an "across variable" and is measured across nodes in the circuit. The sum of across variables around any loop in a circuit equals zero.

I is a "through variable" and is measured in series with a component. The sum of through variables flowing into and out of any node equals zero.

Components in parallel have the same value for $\mathcal{V}$ and the component values for I (loop currents) sum together. Components in series have the same value for I and the component values for $\mathcal{V}$ (voltage drops) sum together.

These are the only rules you should use when laying out a circuit model for a mechanical or acoustic system. Due to the counter-intuitive choice for across and through variables, it is very easy to make a mistake.

Circuit components have the following relationships:

$$
\begin{aligned}
\mathcal{V}(t) &= R\,I(t) \quad \text{or} \quad \mathcal{V}(S) = R\,I(S) \\
\mathcal{V}(t) &= L\tfrac{dI}{dt} \quad \text{or} \quad \mathcal{V}(S) = SL\,I(S) \\
I(t) &= C\tfrac{d\mathcal{V}}{dt} \quad \text{or} \quad \mathcal{V}(S) = \tfrac{I(S)}{SC}
\end{aligned}
$$

where $\mathcal{V}(S)$ and $I(S)$ are the Laplace transforms of $\mathcal{V}(t)$ and $I(t)$.

Note that power equals $\mathcal{V}I = \mathcal{V}^2 R = I^2/R$.

Electrical Systems

$\mathcal{V}$ is voltage (volts).
I is current (amps).
R is resistance (ohms).
L is inductance (henries).
C is capacitance (farads).

Mechanical Systems

$\mathcal{V}$ is velocity v.
I is force f.
R is inverse of viscous damping Q.
L is inverse of spring constant k.
C is mass M.

$(B\ell)^2$ conversion factor

Viscous damping is used by shock absorbers and is described by $f = Qv$. From the spring law $f = kx$ we get $df/dt = kv$. Newton's second law of motion gives us $f = Ma = M\,dv/dt$. Speaker specs include mechanical resistance of unknown origin. Possible source include friction, inelastic deformation and eddy currents.

If $\mathcal{V}$ is given in meters/sec and I in Newtons, L would be in units of meters/Newton and C in kilograms. R should be in units of sec/kilogram (inverse of Kg/sec).

Acoustic Systems

$\mathcal{V}$ is flow rate (Av).
I is pressure $(P - P_0)$.
R is inverse of acoustic damping.
L is compliance (volume).
C is inertia (length/area).

$V/L \approx 1.42 \times 10^5$ Newtons/m^2
$CA/\ell \approx 1.2$ kg/m^3
$(B\ell/A)^2$ conversion factor

If $\mathcal{V}$ is given in m^3/sec and I in Newtons/m^2, then L will be given in m^5/Newton and C in kg/m^4. Both the density of air and the speed of sound vary with temperature and humidity.

The ideal gas law gives us $PV = MkT$ where P is pressure, V is volume, M is the mass, T is the temperature in Kelvin and k is Boltzmann's constant. For a cavity of volume V being fed by flow rate Av we have

$$V\frac{dP}{dt} = kT\frac{dM}{dt} + Mk\frac{dT}{dP}\frac{dP}{dt}$$

where $dM/dt = \rho Av$ and ρ is the density of air. From the adiabatic relationships ($C7$), one can show that

$$\frac{dT}{dP} = \frac{(1-\gamma)V}{Mk}$$

where $\gamma = 1/(1 + k/\mu)$ and μ is the specific heat for air. Substituting this back into the previous equations yields

$$\frac{dP}{dt} = \frac{\rho kT}{\gamma V}Av = \frac{\rho c^2}{V}Av$$

where c is the speed of sound. The equivalent impedance is therefore given by

$$L \doteq \frac{V}{\rho_0 c^2} \tag{A1}$$

where one can use $\rho c^2 \approx 1.42 \times 10^5$ Newtons/m^2.

A tube of length ℓ and cross-section area A contains a mass of $M = \rho\ell A$. Applying a force to this mass, from Newton's second law we get

$$\begin{aligned} \Delta PA &= \ell A\frac{d}{dt}\rho v \\ &= \ell\rho\frac{d}{dt}Av \end{aligned}$$

assuming the velocity is constant along the length of the tube. Dividing both sides by A, we see that

$$\Delta P = \frac{\ell\rho}{A}\frac{d}{dt}Av$$

and the equivalent capacitance is given by

$$C \doteq \rho\frac{\ell}{A} \tag{A2}$$

where one can use $\rho \approx 1.2\,\text{kg/m}^3$.

When the cross-sectional area is not constant, one can use

$$C \approx \frac{\rho V}{A_1 A_2}$$

where V is the volume and A_1 and A_2 are the areas of the two openings. A rigorous treatment would require solving the distributed parameter system.

It should be noted that tubes also exhibit induction (volume) and resistance (air friction), depending on their size. For example, a tube 1 inch in diameter and 4 inches long ($V = \pi$ in^3) has the equivalent C value as a tube 2 inches in diameter and 16 inches long ($V = 16\pi$ in^3) or a tube 1/2 inch in diameter and 1 inch long ($V = \frac{1}{16}\pi$ in^3). Forcing the same amount of air through a smaller diameter will increase the velocity, and air friction is proportional to velocity squared (highly nonlinear). Basically one should avoid excessively large or small tubes, relative to the amount of air flowing through them.

Sound absorbing materials are described by their Q value where

$$Q = \frac{\omega L}{R}$$

and therefore the equivalent resistance is given by

$$R = \frac{\omega L}{Q} = \frac{\omega V}{\rho c^2 Q} \quad . \qquad A3$$

where V is the volume of material used. Since these materials are always located inside enclosures, the equivalent resistor is in parallel with its associated inductor.

Sound absorbing materials are frequency dependent, but since their primary purpose is to deaden resonances, the important value is at the resonance frequency.

Combined Systems

The force generated by a coil in a uniform magnetic field is given by

$$\vec{f} = I \oint (d\vec{\ell} \times \vec{B}) \quad \text{or} \quad f = I\ell B$$

where ℓ is the length of the coil and B is the magnetic field strength. The scalar version assumes that all the vectors are lined up properly. The voltage resulting from moving a coil in a uniform magnetic field is given by

$$\mathcal{V} = \oint (\vec{v} \times \vec{B}) \cdot d\ell \quad \text{or} \quad \mathcal{V} = v\ell B$$

where v is the velocity of the coil. The result of these two equations is that

$$\frac{\mathcal{V}}{I} = (\ell B)^2 \frac{v}{f} \quad . \qquad (A4)$$

The fact that force is a function of current and voltage is function of velocity explains the somewhat counter-intuitive choice for mechanical across and through variables.

To combine electrical and mechanical systems together, one can scale all the values for R, L and C in the mechanical system (multiply for R and L and divide for C) by the factor $(\ell B)^2$. If ℓ is given in meters and B in Teslas, we have

$$1 \text{ meter } \times 1 \text{ Tesla } = 1 \text{ Newton/Amp } = 1 \text{ Volt sec/meter}$$

and consequently

$$1 \text{ Newton/Amp } \times 1 \text{ Volt sec/meter } = 1 \text{ kg/Farad } = 1 \text{ Henry Newton/meter}$$

since

$$\begin{aligned} 1 \text{ Newton} &= 1 \text{ kg meter/sec}^2 \\ 1 \text{ Farad} &= 1 \text{ Amp sec/Volt} \\ 1 \text{ Henry} &= 1 \text{ Volt sec/Amp} \quad . \end{aligned}$$

The relationship between mechanical and acoustic systems is given by

$$\frac{v}{f} = \left(\frac{1}{A^2}\right)\frac{Av}{P-P_0} \tag{A5}$$

where P is the pressure and A is the area of the diaphragm which connects the two systems. Models for the acoustic properties on both sides of a diaphragm are connected in parallel, since both $\mathcal{V}$ and I have opposite signs on opposite sides.

To combine acoustic systems with combined electrical-mechanical systems, one should scale all the values for R, L and C in the acoustic system (multiply for R and L and divide for C) by the factor $(\ell B)^2/A^2$.

Thiele Units

Speakers are sometimes described using the following parameters. The damping factor Q_{ts} is based on measurements of input impedance of the speaker at its resonance frequency. Q_{es} is calculated using

$$Q_{\mathrm{es}} = \frac{\omega_s M_{\mathrm{ms}}}{(B\ell)^2} R_e$$

where $\omega_s = 2\pi f_s$ is the resonance frequency, and Q_{ms} can be found using

$$\frac{1}{Q_{\mathrm{ms}}} + \frac{1}{Q_{\mathrm{es}}} = \frac{1}{Q_{\mathrm{ts}}} \quad .$$

The important thing to us is that one can compute R_m using

$$R_m = \frac{Q_{\mathrm{ms}}}{Q_{\mathrm{es}}} R_e \quad . \tag{A6}$$

since all of these are based on electrical measurements.