Bessel Function Calculations

John Kormylo

Bessel functions are usually calculated using power series approximations near $x = 0$ and by the asymptotic solutions

$$J_0(x) \to \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4}) \tag{1}$$

and

$$J_1(x) \to \sqrt{\frac{2}{\pi x}} \sin(x - \frac{\pi}{4}) \tag{2}$$

as $x \to \infty$. The goal here is to find better approximations over the entire range of x.

Bessel Functions of the First Kind

Bessel functions can be defined as the solution to the second order differential equation

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0 \tag{3}$$

for any n. We really only need to solve for $J_0(x)$ and $J_1(x)$ and use the recursion

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \tag{4}$$

and the identity

$$J_{-n}(x) = (-1)^n J_n(x) \tag{5}$$

for $n = 1, 2, 3, \ldots$.

One can convert this second order differential equations to a first order state vector differential equation where the state vector consists of J_0 and J_1 by using the identities

$$J_0'(x) = -J_1(x) \tag{6}$$

and

$$x J_1'(x) + J_1(x) = x J_0(x) \tag{7}$$

with initial condition $J_0(0) = 1$ and $J_1(0) = 0$.

Multiple Power Series

Given a power series approximation at x_0, one can compute $J_0(x_0 + \Delta x)$ and and $J_1(x_0 + \Delta x)$ for $x_0 = 0, \Delta x, 2\Delta x, \ldots$. By storing these values one can re-construct the corresponding power series to interpolate $J_0(x)$ and $J_1(x)$ for any x within the appropriate range.

Define power series approximations of the form

$$J_0(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m \tag{8}$$

and

$$J_1(x) = \sum_{m=0}^{\infty} b_m (x - x_0)^m \tag{9}$$

for some x_0. Substituting these series into equation (6) gives us

$$\sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1} = -\sum_{m=0}^{\infty} b_m (x - x_0)^m$$

and equivalencing over powers of $(x - x_0)$ yields

$$a_m = -b_{m-1}/m \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{10}$$

starting from $a_0 = J_0(x_0)$.

From equation (7) we can write

$$(x - x_0)J_1'(x) + x_0 J_1'(x) + J_1(x) = (x - x_0)J_0(x) + x_0 J_0(x) \quad .$$

Substituting the power series approximations into this gives us

$$\sum_{m=0}^{\infty}(m+1)b_m(x-x_0)^m + x_0\sum_{m=1}^{\infty} m b_m (x-x_0)^{m-1} = \sum_{m=0}^{\infty} a_m(x-x_0)^{m+1} + x_0\sum_{m=0}^{\infty} a_m(x-x_0)^m \tag{11}$$

and equivalencing over powers of $(x - x_0)$ yields the recursive relationship

$$b_{m+1} = \frac{x_0 a_m + a_{m-1} - (m+1)b_m}{x_0(m+1)} \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{12}$$

starting from $b_1 = a_0 - b_0/x_0$ and $b_0 = J_1(x_0)$.

When $x_0 = 0$, equation (11) simplifies to

$$\sum_{m=0}^{\infty}(m+1)b_m x^m = \sum_{m=0}^{\infty} a_m x^{m+1}$$

and equivalencing over powers of x gives us

$$b_m = a_{m-1}/(m+1) \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{13}$$

starting from $a_0 = 1$ and $b_0 = 0$. Solving for a_m and b_m recursively using (10) and (13) will produce the known closed-form solution for the power series coefficients, but it turns out that the recursive solution is more numerically robust.

Bessel Functions of the Second Kind

These functions are defined as

$$Y_n(x) = (\alpha \log x + \beta)J_n(x) + x^{-n} f_n(x) \tag{14}$$

where (somewhat arbitrarily) $\alpha = 2/\pi$ and $\beta = 0.57735 - \log 2$. Inserting this into the identity

$$Y_0'(x) = -Y_1(x)$$

and using (6) gives us

$$\alpha J_0(x) + x f_0'(x) = -f_1(x) \tag{15}$$

while the identity

$$xY_1'(x) + Y_1(x) = xY_0(x)$$

and (7) gives us

$$\alpha J_1(x) + f_1'(x) = x f_0(x) \tag{16}$$

which we will use to solve for $f_0(x)$ and $f_1(x)$.

Define power series approximations of the form

$$f_0(x) = \sum_{m=0}^{\infty} c_m (x - x_0)^m \tag{17}$$

and

$$f_1(x) = \sum_{m=0}^{\infty} d_m (x - x_0)^m \tag{18}$$

for some x_0. Substituting (8), (17) and (18) into (15) gives us

$$\alpha \sum_{m=0}^{\infty} a_m (x - x_0)^m + \sum_{m=1}^{\infty} m c_m (x - x_0)^m + x_0 \sum_{m=1}^{\infty} m c_m (x - x_0)^{m-1} = -\sum_{m=0}^{\infty} d_m (x - x_0)^m$$

and equivalencing over powers of $(x - x_0)$ yields the recursion

$$c_{m+1} = -\frac{m c_m + d_m + \alpha a_m}{x_0 (m+1)} \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{19}$$

starting from $c_1 = -(d_0 + \alpha a_0)/x_0$ and $c_0 = f_0(x_0)$. (We assume a_m and $b_m \forall m$ have already been computed.) When $x_0 = 0$, we instead get

$$c_m = -\frac{d_m + \alpha a_m}{m} \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{20}$$

starting from $c_0 = 0$.

Substituting (9), (17) and (18) into (16) gives us

$$\alpha \sum_{m=0}^{\infty} b_m (x - x_0)^m + \sum_{m=1}^{\infty} m d_m (x - x_0)^{m-1} = \sum_{m=0}^{\infty} c_m (x - x_0)^{m+1} + x_0 \sum_{m=0}^{\infty} c_m (x - x_0)^m$$

and equivalencing over powers of $(x - x_0)$ yields

$$d_{m+1} = \frac{x_0 c_m + c_{m-1} - \alpha b_m}{(m+1)} \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{22}$$

starting from $d_1 = x_0 c_0 - \alpha b_0$ and $d_0 = f_1(x_0)$. This also works when $x_0 = 0$, except that we start from $d_0 = -\alpha$.

Trigonometric Hybrid

Assume a solution of the form

$$J_0(x) = a(x)\cos(x) + b(x)\sin(x) \tag{23}$$

and

$$J_1(x) = a(x)\sin(x) - b(x)\cos(x) \tag{24}$$

which correspond more closely to the asymptotic solutions. In fact, from (2) and (3) one can show that these are given by

$$a(x) \to \frac{1}{\sqrt{\pi x}} \quad \text{and} \quad b(x) \to \frac{1}{\sqrt{\pi x}}$$

as $x \to \infty$.

The inverse relationship, as derived in the Appendix, can be written as

$$a(x) = \cos(x) J_0(x) + \sin(x) J_1(x) \tag{25}$$

and

$$b(x) = \sin(x) J_0(x) - \cos(x) J_1(x) \quad . \tag{26}$$

However, we intend to solve for $a(x)$ and $b(x)$ directly using differential equations starting from $a(0) = 1$ and $b(0) = 0$.

Substituting (23) and (24) into (6) gives us

$$a'(x)\cos(x) - a(x)\sin(x) + b'(x)\sin(x) + b(x)\cos(x) = -a(x)\sin(x) + b(x)\cos(x)$$

which simplifies to

$$a'(x)\cos(x) + b'(x)\sin(x) = 0 \tag{27}$$

while substituting (23) and (24) into (7) gives us

$$\begin{aligned} x\big(a'(x)\sin(x) + a(x)\cos(x) - b'(x)\cos(x) + b(x)\sin(x)\big) & \\ + a(x)\sin(x) - b(x)\cos(x) &= x\big(a(x)\cos(x) + b(x)\sin(x)\big) \end{aligned}$$

which simplifies to

$$a'(x)\sin(x) - b'(x)\cos(x) = -a(x)\frac{\sin(x)}{x} + b(x)\frac{\cos(x)}{x} \tag{28}$$

for $x \neq 0$. Combining (27) and (28) gives us

$$2xa'(x) = \big(\cos(2x) - 1\big)a(x) + \sin(2x)b(x) \tag{29}$$

and

$$2xb'(x) = \sin(2x)a(x) - \big(\cos(2x) + 1\big)b(x) \tag{30}$$

(derived in the Appendix).

If order to solve for $a(x)$ and $b(x)$ we will equivalence the high order derivatives when $x = x_0$. From (29) one can show that the m^{th} derivative is given by

$$2xa^{(m+1)}(x) + (2m+1)a^{(m)}(x) = \sum_{i=0}^{m} \left(\frac{m!}{i!(m-i)!}\right) 2^i \big(c_{i+1}(2x)a^{(m-i)}(x) + c_i(2x)b^{(m-i)}(x)\big) \tag{31}$$

and from (30) we get

$$2xb^{(m+1)}(x) + (2m+1)b^{(m)}(x) = \sum_{i=0}^{m} \left(\frac{m!}{i!(m-i)!}\right) 2^i \big(c_i(2x)a^{(m-i)}(x) + c_{i-1}(2x)b^{(m-i)}(x)\big) \tag{32}$$

for $m = 0, 1, 2, \dots$ where $c_m(x)$ is defined as

$$c_m(x) = \begin{cases} \sin(x) & : m \bmod 4 = 0 \\ \cos(x) & : m \bmod 4 = 1 \\ -\sin(x) & : m \bmod 4 = 2 \\ -\cos(x) & : m \bmod 4 = 3 \end{cases} .$$

At this point let us replace $a(x)$ and $b(x)$ by power series of the form

$$a(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

and

$$b(x) = \sum_{m=0}^{\infty} b_m (x - x_0)^m \quad .$$

Substituting these into (31) when $x = x_0$ we get

$$2x_0(m+1)!\,a_{m+1} + (2m+1)m!\,a_m = \sum_{i=0}^{m} \left(\frac{m!}{i!}\right) 2^i \big(c_{i+1}(2x_0)a_{m-i} + c_i(2x_0)b_{m-i}\big) \tag{33}$$

while from (32) we get

$$2x_0(m+1)!\,b_{m+1} + (2m+1)m!\,b_m = \sum_{i=0}^{m} \left(\frac{m!}{i!}\right) 2^i \big(c_i(2x_0)a_{m-i} + c_{i-1}(2x_0)b_{m-i}\big) \quad . \tag{34}$$

which can be used to recursively generate a_{m+1} and b_{m+1} for $m = 0, 1, 2, \ldots$ starting from $a_0 = a(x_0)$ and $b_0 = b(x_0)$ for any $x_0 \neq 0$.

When $x_0 = 0$ from (33) we get

$$(2m)m!\,a_m = \sum_{i=1}^{m} \left(\frac{m!}{i!}\right) 2^i \big(c_{i+1}(0)a_{m-i} + c_i(0)b_{m-i}\big) \tag{35}$$

and from (34) we get

$$(2m+2)m!\,b_m = \sum_{i=1}^{m} \left(\frac{m!}{i!}\right) 2^i \big(c_i(0)a_{m-i} + c_{i-1}(0)b_{m-i}\big) \tag{36}$$

which can be used to recursively generate a_m and b_m for $m = 1, 2, 3, \ldots$ starting from $a_0 = 1$ and $b_0 = 0$.

Appendix - Matrix Equations

Equations (23) and (24) can be written in matrix form as

$$\begin{bmatrix} \cos(x) & \sin(x) \\ \sin(x) & -\cos(x) \end{bmatrix} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix} = \begin{bmatrix} J_0(x) \\ J_1(x) \end{bmatrix}$$

which gives us

$$\begin{bmatrix} a(x) \\ b(x) \end{bmatrix} = \begin{bmatrix} \cos(x) & \sin(x) \\ \sin(x) & -\cos(x) \end{bmatrix} \begin{bmatrix} J_0(x) \\ J_1(x) \end{bmatrix}$$

since

$$\begin{bmatrix} \cos(x) & \sin(x) \\ \sin(x) & -\cos(x) \end{bmatrix} \begin{bmatrix} \cos(x) & \sin(x) \\ \sin(x) & -\cos(x) \end{bmatrix} = I$$

(the matrix is its own inverse).

Equations (27) and (28) can be written in matrix form as

$$\begin{bmatrix} \cos(x) & \sin(x) \\ \sin(x) & -\cos(x) \end{bmatrix} \begin{bmatrix} a'(x) \\ b'(x) \end{bmatrix} = \frac{1}{x} \begin{bmatrix} 0 & 0 \\ -\sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix}$$

which gives us

$$\begin{bmatrix} a'(x) \\ b'(x) \end{bmatrix} = \frac{1}{x} \begin{bmatrix} -\sin^2(x) & \cos(x)\sin(x) \\ \cos(x)\sin(x) & -\cos^2(x) \end{bmatrix} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix}$$

which can also be written as

$$2x \begin{bmatrix} a'(x) \\ b'(x) \end{bmatrix} = \begin{bmatrix} \cos(2x) - 1 & \sin(2x) \\ \sin(2x) & -\cos(2x) - 1 \end{bmatrix} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix} \quad .$$