Deconvolution and Wavelet Estimation

John Kormylo

Here we are concerned with converting the autocovariance function into a convolution of two wavelets, or more precisely, one wavelet and its anti-causal counterpart. In matrix form this can be written as

$$C = UU' \tag{1}$$

where C is the covariance matrix and U is both upper triangular and Toeplitz. If U is given by

$$U = \begin{bmatrix} h(0) & h(1) & h(2) & \dots & h(N-1) \\ 0 & h(0) & h(1) & \dots & h(N-2) \\ 0 & 0 & h(0) & \dots & h(N-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h(0) \end{bmatrix} \tag{2}$$

then

$$c_{ij} = \sum_{k=1}^{N} h(k-i)h(k-j) \tag{3}$$

where $h(k)$ for $k = 0, 1, \dots, N-1$ is the wavelet we wish to obtain.

The Cholevski algorithm will factor a covariance matrix into triangular matrices, but they are not Toeplitz. The Levinson algorithm generates a Toeplitz matrix, but it is an inverse, not a factor. One thing that will work is to take a Fourier transform, take the logarithm, take an inverse Fourier transform, remove the negative time components and reduce the zero time component by half, take a Fourier transform, take the complex exponential, and take an inverse Fourier transform. Of course, there are numerical difficulties from taking the logarithm of very small numbers.

The iterative algorithm presented here is the same as that used for computing square roots, more or less. Assume an estimate $U(k)$ with its corresponding error matrix $E(k)$ so that

$$\begin{aligned} C &= \big[U(k) + E(k)\big] \, \big[U(k) + E(k)\big]' \\ &= U(k)U'(k) + U(k)E'(k) + E(k)U'(k) + E(k)E'(k) \end{aligned}$$

where both $U(k)$ and $E(k)$ are constrained to be upper triangular and Toeplitz. If the error is small compared to the estimate, then

$$C - U(k)U'(k) \approx U(k)E'(k) + E(k)U'(k) \tag{4}$$

and we can compute $E(k)$ to obtain a new estimate using

$$U(k+1) = U(k) + E(k) \quad . \tag{5}$$

From now on we will simply assume we are dealing with the k^{th} iteration and drop the explicit notation. Since E is Toeplitz, we need only compute the first column of E', so let us define $\mathbf{e}$ as the first column of E' and $\mathbf{c}$ as the first column of $(C - UU')$.

It should be noted that while UE' and EU' are both Toeplitz, they are neither triangular nor symmetrical. If they were triangular, one could use $\mathbf{e} = U^{-1}\mathbf{c}$, which incidentally does not work. If they were symmetrical, one could use $\mathbf{e} = \frac{1}{2}U^{-1}\mathbf{c}$, which incidentally converges reliably albeit slowly.

In fact, one can show that the $\mathbf{c}$ is given by

$$\mathbf{c} = U\mathbf{e} + H\mathbf{e} = [U + H]\mathbf{e} \tag{6}$$

where

$$H = \begin{bmatrix} h(0) & h(1) & h(2) & \dots & h(N-1) \\ h(1) & h(2) & h(3) & \dots & 0 \\ h(2) & h(3) & h(4) & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ h(N-1) & 0 & 0 & \dots & 0 \end{bmatrix} \quad . \tag{7}$$

Computing $\mathbf{e}$ using (6) converges reliably and rapidly, although it does require inverting a rather large matrix.

It should be noted that, based on autocovariance data, one cannot obtain a unique solution for the wavelet. While this algorithm has no explicit minimum phase criterion, it does constrain $h(0)$ in that the determinant of covariance matrix C equals $[h(0)]^{2N}$.

The addition of observation noise will increase the main diagonal of C by variance R. If one uses the estimate $R = C_{11} - h^2(0)$ then $c_1 = 0$ in (6) every interation.