GK Matrix Reduction

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Gaussian reduction is used to compute matrix determinants, inverses and LU decompositions. GK reduction performs the same function, except that it will work on algebras not closed under division (such as integers or polynomials). For example, it can be used to compute the characteristic polynomial for a matrix, or to handle circuit problems using floating nodal admitance matrices.

The iterative algorithm for an arbitrary $n \times n$ matrix A is given by

$$A(k+1)_{i,j} = \frac{A(k)_{i,j} A(k)_{k,k} - A(k)_{i,k} A(k)_{k,j}}{d(k)} \quad \forall \quad i,j > k$$

for $k = 1, 2, \ldots, n$ where $d(k)$ is chosen to divide evenly for every i, j, starting from $A(1) = A$ and $d(1) = 1$. Note that setting $d(k) = A(k)_{k,k}$ would reduce to Gaussian reduction (but would not divide evenly).

The clever bit is to realize that

$$d(k+1) = A(k)_{k,k} \quad \text{for} \quad k = 1, 2, \ldots, n$$

will always divide evenly. This can be proven for an arbitrary 3×3 matrix over 2 iterations (ugly but straight foreward), then extended to any size matrix over any number of iterations by induction.

It should be noted that $d(k+1)$ is in fact the determinant of the first $k \times k$ submatrix of A. Furthermore, $A(k)$ produced by GK reduction equals $d(k)$ times the matrix which would have been produced by Gaussian reduction. Consequently the same techniques used to compute the matrix inverse using Gaussian reduction will also work with GK reduction, except that the result still needs to be divided by the determinant.

This should be expected since the matrix inverse can be written as

$$A^{-1}_{i,j} = (-1)^{i+j} M_{j,i} / \det A$$

where M is the matrix of minors of A (determinants of corresponding submatrices of A with one row and one column removed).