Solving Integrals using Differential Equations

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In acoustics one often finds integrals of the form

$$F(\omega/c) = \int_0^{\omega/c} f(k_r)\, g(k_z)\, dk_z$$

or

$$G(\omega/c) = \int_0^{\omega/c} f(k_r)\, g(k_z) k_r\, dk_r$$

where

$$k_r^2 + k_z^2 = \left(\frac{\omega}{c}\right)^2$$

and where $f(x)$ and $g(x)$ generally satisfy ordinary differential equations. While there may not be a closed form solution, it may be possible to find differential equations which are satisfied by $F(x)$ and $G(x)$.

Using the substitutions

$$x = \frac{\omega}{c} \quad , \quad k_r = x \sin\theta \quad \text{and} \quad k_z = x\cos\theta$$

we can rewrite our integrals as

$$F(x) = x \int_0^{\pi/2} f(x\sin\theta)\, g(x\cos\theta) \sin\theta\, d\theta \tag{1}$$

and

$$G(x) = x^2 \int_0^{\pi/2} f(x\sin\theta)\, g(x\cos\theta) \sin\theta \cos\theta\, d\theta \tag{2}$$

to satisfy the constraint on k_r and k_z.

Solving for $F(x)$

Define state vector $\mathbf{v}(x)$ using

$$v_1(x) = x \int_0^{\pi/2} f(x\sin\theta)\, g(x\cos\theta) \sin\theta\, d\theta \tag{3}$$

$$v_2(x) = x \int_0^{\pi/2} f(x\sin\theta)\, g(x\cos\theta) \cos\theta\, d\theta \tag{4}$$

and

$$v_3(x) = \int_0^{\pi/2} f(x\sin\theta)\, g(x\cos\theta)\, d\theta \tag{5}$$

and solve for the state vector model

$$\mathbf{v}'(x) = A(x)\mathbf{v}(x) + \mathbf{b}(x) \tag{6}$$

$$F(x) = \mathbf{h}^\top \mathbf{v}(x) \tag{7}$$

where, in this case,

$$\mathbf{h}^\top = (\,1 \quad 0 \quad 0\,) \quad .$$

An alternative formulation is to include $f(x)$ and/or $g(x)$ in an expanded state vector instead of putting them into $\mathbf{b}(x)$, and solving for both at the same time.

As will be shown later, one can obtain the equations

$$v_1'(x) = f(0)\,g(x) - f(x)\lim_{y\to 0} yg(xy) + x\int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\,d\theta \tag{8}$$

$$v_2'(x) = f(x)\,g(0) - g(x)\lim_{y\to 0} yf(xy) + x\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\,d\theta \tag{9}$$

and

$$v_3'(x) = \int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\sin\theta\,d\theta + \int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\cos\theta\,d\theta \tag{10}$$

by taking partial derivatives w.r.t. x and using integration by parts. When f and g satisfy first order differential equations, one can substitute for f' and g' in (8) through (10) and obtain integrals corresponding to other states in the vector. Then it is a matter of putting the appropriate terms into matrix $A(x)$ and vector $\mathbf{b}(x)$.

When $f(x)$ and $g(x)$ satisfy higher order o.d.e.s one must construct state vector models and define $\mathbf{v}(x)$ to include the three integrals for each of these model states.

Solving for $G(x)$

Define state vector $\mathbf{v}(x)$ using (3) and (4) as before, plus

$$v_3(x) = x^2\int_0^{\pi/2} f(x\sin\theta)\,g(x\cos\theta)\sin(2\theta)\,d\theta \tag{11}$$

and

$$v_4(x) = x^2\int_0^{\pi/2} f(x\sin\theta)\,g(x\cos\theta)\cos(2\theta)\,d\theta \tag{12}$$

where, in this case,

$$G(x) = (\,0 \quad 0 \quad \tfrac{1}{2} \quad 0\,)\,\mathbf{v}(x) \quad .$$

As will be shown later, one can obtain the equations

$$\begin{aligned} xv_1'(x) - 2v_1(x) = x\big(f(x)\lim_{y\to 0} yg(xy) - f(0)\,g(x)\big) \\ - x^2\int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\cos(2\theta)\,d\theta \\ + x^2\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\sin(2\theta)\,d\theta \end{aligned} \tag{13}$$

$$
\begin{aligned}
x v_2'(x) - 2 v_2(x) &= x\big(g(x) \lim_{y\to 0} y f(xy) - f(x)\, g(0)\big) \\
&\quad + x^2 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta) \sin(2\theta)\, d\theta \\
&\quad + x^2 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta) \cos(2\theta)\, d\theta \qquad (14) \\
v_3'(x) &= x\big(f(0)\, g(x) - f(x)\, g(0)\big) \\
&\quad + x^2 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta) \cos\theta\, d\theta \\
&\quad + x^2 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta) \sin\theta\, d\theta \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
v_4'(x) &= x\big(f(x) \lim_{y\to 0} y g(xy) - g(x) \lim_{y\to 0} y f(xy)\big) \\
&\quad + x^2 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta) \sin\theta\, d\theta \\
&\quad + x^2 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta) \cos\theta\, d\theta \qquad (16)
\end{aligned}
$$

from which one can construct $A(x)$ and $\mathbf{b}(x)$ as before.

Example

Let $f(k_r) = \exp(jrk_r)$ and $g(k_z) = \exp(jzk_z)$, which satisfy the differential equations

$$
f'(x) = jrf(x) \quad \text{and} \quad g'(x) = jzg(x) \quad .
$$

(Using complex numbers effectively turns a second order o.d.e. into a state vector model.) Substituting these into equations (8) through (10) we get

$$
\begin{aligned}
v_1'(x) &= e^{jzx} + jrx\, v_3(x) \\
v_2'(x) &= e^{jrx} + jzx\, v_3(x)
\end{aligned}
$$

and

$$
v_3'(x) = \frac{jr}{x} v_1(x) + \frac{jz}{x} v_2(x)
$$

respectively, and therefore

$$
A(x) = \begin{pmatrix} 0 & 0 & jrx \\ 0 & 0 & jzx \\ \frac{jr}{x} & \frac{jz}{x} & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b}(x) = \begin{pmatrix} e^{jzx} \\ e^{jrx} \\ 0 \end{pmatrix} \quad .
$$

Since $A(x)$ has a rank of only 2, this model corresponds to a second order differential equation. From (6) and (7) one can show that

$$
F(x) = \mathbf{h}^\top \left(\frac{\partial}{\partial x} I - A(x) \right)^{-1} \mathbf{b}(x)
$$

and therefore

$$F''(x) + (r^2 + z^2)F(x) = jze^{jrx}$$

since

$$\left(\frac{\partial}{\partial x} I - A(x)\right)^{-1} = \frac{1}{\left(\frac{\partial^2}{\partial x^2} + r^2 + z^2\right)\frac{\partial}{\partial x}} \begin{pmatrix} \frac{\partial^2}{\partial x^2} + z^2 & -rz & \frac{\partial}{\partial x} jrx \\ -rz & \frac{\partial^2}{\partial x^2} + r^2 & \frac{\partial}{\partial x} jzx \\ \frac{\partial}{\partial x}\frac{jr}{x} & \frac{\partial}{\partial x}\frac{jz}{x} & \frac{\partial^2}{\partial x^2} \end{pmatrix} \quad .$$

The extra $\partial/\partial x$ in the determinant was used to integrate $(-rz)\exp(jrx)$.

Derivations

Taking the partial of (5) w.r.t. x gives us (10) directly.
Taking the partial of (3) w.r.t. x gives us

$$\begin{aligned} v_1'(x) = & \int_0^{\pi/2} f(x\sin\theta)\, g(x\cos\theta) \sin\theta \\ & + x \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta)\sin^2\theta \, d\theta \\ & + x \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta)\sin\theta\cos\theta\, d\theta \end{aligned}$$

and therefore, multiplying by x and subtracting (3),

$$\begin{aligned} xv_1'(x) - v_1(x) = & \, x^2 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta)\sin^2\theta\, d\theta \\ & + x^2 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta)\sin\theta\cos\theta\, d\theta \quad . \end{aligned} \tag{17}$$

Using integration by parts on (3) where $dv = \sin\theta\, d\theta$ gives us

$$\begin{aligned} v_1(x) = & \, x\big(f(0)\, g(x) - f(x) \lim_{y\to 0} yg(xy)\big) \\ & + x^2 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta)\cos^2\theta\, d\theta \\ & - x^2 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta)\sin\theta\cos\theta\, d\theta \quad . \end{aligned} \tag{18}$$

Adding (18) to (17) gives us

$$\begin{aligned} xv_1'(x) = & \, x\big(f(0)\, g(x) - f(x)\lim_{y\to 0} yg(xy)\big) \\ & + x^2 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta)\, d\theta \end{aligned}$$

and dividing both sides by x yields (8). Subtracting (18) from (17) yields (13).

Taking the partial of (4) w.r.t. x gives us

$$\begin{aligned} v_2'(x) = & \int_0^{\pi/2} f(x\sin\theta)\,g(x\cos\theta)\cos\theta \\ & + x\int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\sin\theta\cos\theta\,d\theta \\ & + x\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\cos^2\theta\,d\theta \end{aligned}$$

and therefore, multiplying by x and subtracting (4),

$$\begin{aligned} xv_2'(x) - v_2(x) = & \; x^2\int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\sin\theta\cos\theta\,d\theta \\ & + x^2\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\cos^2\theta\,d\theta \quad . \end{aligned} \tag{19}$$

Using integration by parts on (4) where $dv = \cos\theta\,d\theta$ gives us

$$\begin{aligned} v_2(x) = & \; x\big(f(x)\,g(0) - g(x)\lim_{y\to 0} yf(xy)\big) \\ & - x^2\int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\sin\theta\cos\theta\,d\theta \\ & + x^2\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\sin^2\theta\,d\theta \quad . \end{aligned} \tag{20}$$

Adding (20) to (19) gives us

$$\begin{aligned} xv_2'(x) = & \; x\big(f(x)\,g(0) - g(x)\lim_{y\to 0} yf(xy)\big) \\ & + x^2\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\,d\theta \end{aligned}$$

and dividing both sides by x yields (9). Subtracting (20) from (19) yields (14).

Taking the partial of (11) w.r.t. x gives us

$$\begin{aligned} v_3'(x) = & \; 2x\int_0^{\pi/2} f(x\sin\theta)\,g(x\cos\theta)\sin(2\theta) \\ & + 2x^2\int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\sin^2\theta\cos\theta\,d\theta \\ & + 2x^2\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\sin\theta\cos^2\theta\,d\theta \end{aligned}$$

and therefore, multiplying by x and subtracting (11) times 2,

$$xv_3'(x) - 2v_3(x) = 2x^3 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta) \sin^2\theta \cos\theta \, d\theta$$
$$+ 2x^3 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta) \sin\theta \cos^2\theta \, d\theta \quad . \tag{21}$$

Multiplying (11) by 2 and using integration by parts where $dv = 2\sin(2\theta)\, d\theta$ gives us

$$2v_3(x) = x^2\big(f(0)\, g(x) - f(x)\, g(0)\big)$$
$$+ x^3 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta) \cos\theta \cos(2\theta) \, d\theta$$
$$- x^3 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta) \sin\theta \cos(2\theta) \, d\theta \quad . \tag{22}$$

Substituting for $\cos(2\theta)$ using

$$\cos(2\theta) = 1 - 2\sin^2\theta = 2cos^2\theta - 1$$

and adding (22) to (21) gives us

$$xv_3'(x) = x^2\big(f(0)\, g(x) - f(x)\, g(0)\big)$$
$$+ x^3 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta) \cos\theta \, d\theta$$
$$+ x^3 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta) \sin\theta \, d\theta$$

and dividing both sides by x yields (15).

Taking the partial of (12) w.r.t. x gives us

$$v_4'(x) = 2x \int_0^{\pi/2} f(x\sin\theta)\, g(x\cos\theta) \cos(2\theta)$$
$$+ 2x^2 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta) \sin\theta \cos(2\theta) \, d\theta$$
$$+ 2x^2 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta) \cos\theta \cos(2\theta) \, d\theta$$

and therefore, multiplying by x and subtracting (12) times 2,

$$xv_4'(x) - 2v_4(x) = x^3 \int_0^{\pi/2} f'(x\sin\theta)\, g(x\cos\theta) \sin\theta \cos(2\theta) \, d\theta$$
$$+ x^3 \int_0^{\pi/2} f(x\sin\theta)\, g'(x\cos\theta) \cos\theta \cos(2\theta) \, d\theta \quad . \tag{23}$$

Multiplying (12) by 2 and using integration by parts where $dv = 2\cos(2\theta)\,d\theta$ gives us

$$\begin{aligned} 2v_4(x) &= x^2\big(f(x)\lim_{y\to 0} yg(xy) - g(x)\lim_{y\to 0} yf(xy)\big) \\ &\quad - 2x^3\int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\sin\theta\cos^2\theta\,d\theta \\ &\quad + 2x^3\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\sin^2\theta\cos\theta\,d\theta \quad . \end{aligned} \tag{24}$$

Substituting for $\cos(2\theta)$ as before and adding (24) to (23) gives us

$$\begin{aligned} xv_4'(x) &= x^2\big(f(x)\lim_{y\to 0} yg(xy) - g(x)\lim_{y\to 0} yf(xy)\big) \\ &\quad + x^3\int_0^{\pi/2} f'(x\sin\theta)\,g(x\cos\theta)\sin\theta\,d\theta \\ &\quad + x^3\int_0^{\pi/2} f(x\sin\theta)\,g'(x\cos\theta)\cos\theta\,d\theta \end{aligned}$$

and dividing both sides by x yields (16).