Computing Square Roots using Sequential Approximation

Let x_n be an approximation to $x = \sqrt{y}$.

Our goal is to find some x_{n+1} which is an even better approximation, so that the sequence $x_0, x_1, \ldots$ converges to $\sqrt{y}$.

Let x_n be an approximation to $x = \sqrt{y}$.

Our goal is to find some x_{n+1} which is an even better approximation, so that the sequence $x_0, x_1, \ldots$ converges to $\sqrt{y}$.

Let e_n be the difference between x_n and $x = \sqrt{y}$, so that

$$
\begin{aligned}
y &= x^2 \\
y &= (x_n + e_n)^2 \\
y &= x_n^2 + 2x_n e_n + e_n^2 \quad .
\end{aligned}
$$

Let x_n be an approximation to $x = \sqrt{y}$.

Our goal is to find some x_{n+1} which is an even better approximation, so that the sequence $x_0, x_1, \ldots$ converges to $\sqrt{y}$.

Let e_n be the difference between x_n and $x = \sqrt{y}$, so that

$$
\begin{aligned}
y &= x^2 \\
y &= (x_n + e_n)^2 \\
y &= x_n^2 + 2x_n e_n + e_n^2 \quad .
\end{aligned}
$$

If $|e_n| < x$ then $e_n^2 \ll y$ and

$$y \approx x_n^2 + 2x_n e_n$$

which has the solution

$$e_n \approx \frac{y - x_n^2}{2x_n} \quad .$$

Our new approximation is chosen as

$$x_{n+1} = x_n + \frac{y - x_n^2}{2x_n}$$

which can also be simplified as

$$x_{n+1} = \frac{y + x_n^2}{2x_n} \quad .$$

Our new approximation is chosen as

$$x_{n+1} = x_n + \frac{y - x_n^2}{2x_n}$$

which can also be simplified as

$$x_{n+1} = \frac{y + x_n^2}{2x_n} \quad .$$

Convergence

Let $a = x_n/x$ so that $x_n = ax$ and

$$x_{n+1} = \frac{y + a^2 y}{2ax} = \frac{1 + a^2}{2a} x$$

and therefore

$$\frac{x_{n+1}}{x} = \frac{1 + a^2}{2a} \quad .$$

Since $e_{n+1} = x - x_{n+1}$ we have

$$\begin{aligned}\frac{e_{n+1}}{x} &= 1 - \frac{x_{n+1}}{x} \\ &= 1 - \frac{1+a^2}{2a} \\ &= \frac{2a - 1 - a^2}{2a} \\ &= \frac{-(1 - 2a + a^2)}{2a} \\ &= \frac{-(1-a)^2}{2a} \quad .\end{aligned}$$

Since $e_{n+1} = x - x_{n+1}$ we have

$$\begin{aligned}\frac{e_{n+1}}{x} &= 1 - \frac{x_{n+1}}{x} \\ &= 1 - \frac{1+a^2}{2a} \\ &= \frac{2a - 1 - a^2}{2a} \\ &= \frac{-(1 - 2a + a^2)}{2a} \\ &= \frac{-(1-a)^2}{2a} \quad .\end{aligned}$$

Since $e_n = (1-a)x$ we see that

$$\frac{e_{n+1}}{e_n} = \frac{a-1}{2a}$$

and therefore $|e_{n+1}| < |e_n|$ for all $a > 1/3$.